Class 14, given on Feb 3, 2010, for Math 13, Winter 2010

Recall that the factor which appears in a change of variable formula when integrating is the Jacobian, which is the determinant of a matrix of first order partial derivatives.

Example. Check that the Jacobian of the transformation to spherical coordinates is $\rho^2 \sin\phi$.

The formulas relating rectangular to spherical coordinates are $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Therefore, the Jacobian is given by

$$\begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}.$$

If one expands this determinant in the usual way, after a lot of gathering terms and using the identity $\cos^2 + \sin^2 = 1$, the above expression eventually equals $\rho^2\sin\phi$.

Here are a few remarks about the Jacobian:

- The Jacobian and change of variable formula is a generalization of u-substitution. If we make a u-substitution $u = u(x)$, then we have a formula
$$\int f(u)\,du = \int f(u(x))u'(x)\,dx.$$
The Jacobian of the transformation $u = u(x)$ is $u'(x)$, which is exactly the factor appearing on the right side of this equation. The missing absolute value sign is accounted for by the fact that if $u'(x)$ is negative, the bounds of integration are interchanged.
- If you have been carefully paying attention to the definition of a Jacobian, you might be somewhat bothered by the fact that the Jacobian seems to depend on the ordering of the variables in each of the two variable systems, since the ordering determines the order of the rows and columns of the matrix in the Jacobian. It turns out that interchanging rows and columns of a matrix may change the sign of a determinant, but never the absolute value of the determinant, so the ordering of the variables does not particularly matter when calculating a Jacobian. You may end up with an answer which differs by a minus sign from someone else with a different ordering, but when using the change of variable formula, you take the absolute value of the Jacobian so the sign ambiguity will disappear.

Here is a typical example of using a change of variables which is not polar, cylindrical, or spherical. You need to first decide which change of coordinates you should use to bring the problem to a manageable form, and then make the appropriate coordinate change. Deciding which change to use is more of an art than a science, and requires a lot of practice to get consistently right. Nevertheless there are some clues which might help you make the correct coordinate change.

Example. Evaluate the double integral

$$\iint_D e^{(y-x)/(y+x)}\,dA.$$

where D is the triangle with vertices $(0,0),(2,0),(0,2)$.

The presence of $y - x, y + x$ in a fraction suggests we should make a change of variable like $u = y - x, v = y + x$. (It is still not obvious at this point that this is the correct change to make.) We start by determining what the corresponding domain of integration, say R, in the uv plane is. The triangle D lies in the xy plane, so to determine R we should look at the image of D under the map $u = y - x, v = y + x$. In particular, we can determine the boundary of R by looking at how this change of variables to uv coordinates acts on the boundary of the triangle D.

For example, the side with endpoints $(0,0),(2,0)$ is given by $0 \leq x \leq 2, y = 0$. The corresponding uv coordinates are then $u = -x, v = x, 0 \leq x \leq 2$, so this side maps to the side $v = -u, -2 \leq u \leq 0$. In particular, in the uv plane this is a line segment with endpoints $(0,0),(-2,2)$. We also find that the other two line segments map to line segments, and that R is actually a triangle with vertices $(0,0),(-2,2),(2,2)$.

Therefore we can describe R using inequalities $0 \leq v \leq 2, -v \leq u \leq v$. We now need to calculate the Jacobian of this transformation. We need to solve for x, y in terms of u, v. Fortunately, in this example this is easy, and we see that $x = (v-u)/2, y = (v+u)/2$. The Jacobian is given by the following determinant:

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -1/2.$$

Therefore, the original integral is equal to the integral

$$\iint\limits_R e^{u/v}(1/2)\,dA = \frac{1}{2}\int_0^2\int_{-v}^{v} e^{u/v}\,du\,dv = \frac{1}{2}\int_0^2 ve^{u/v}\Big|_{u=-v}^{u=v}\,dv = \frac{1}{2}\int_0^2 ve - ve^{-1} = e - 1/e.$$

1. Line integrals

We have spent a few weeks talking about higher-dimensional generalizations of definite integrals, and discussed how to calculate them, as well as various applications of double and triple integrals in real life. We now try to generalize the notion of an integral in a different direction. Instead of focusing on defining integrals over high dimensional objects, we will develop a theory of how to integrate not over just an interval on the real axis, but over curves.

Let's motivate this idea with a whimsical example. The American artist Richard Serra is known for his minimalist sculptures, many of which are curving sheets of metal. If you were working for Richard Serra a question you might encounter is just how much sheet metal you need to make one of his installations. In more mathematically precise language, suppose an installation of sheet metal is going to be placed over a curve C in the xy plane. For example, you might be given C using parametric equations $x = x(t), y = y(t), a \leq t \leq b$. Furthermore, Serra has told you what the height of the installation is at any point of C. This might be given to you by a function $f(x,y)$, defined on C. How much sheet metal (that is, what is the surface area) is needed for the sculpture?

One the one hand, if C were a line segment; say a line segment in the x-axis, then we could just integrate f over that line segment in the x-axis, since that integral is the area under the height function f, which gives the surface area of the installation. On the other hand, if $f(x,y)$ were constant, then the surface area of the metal would be the length of C times the height $f(x,y)$. The length of C is the arc length of C, which we know how to calculate: the arc length is equal to the integral

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

The situation we're dealing with is a mix of these two simpler cases: we are allowing the function $f(x,y)$ to vary, while letting C be a curve in the xy plane. In any case, this seems to be the type of problem integration is suited for.

The key idea behind integration is that it is the limit of approximations made by assuming f is constant over various pieces of the region we are integrating over. Suppose we make an approximation to the surface area of the metal as follows. We select various points of C, say $P_0, P_1, \ldots, P_n$, where P_0, P_n are the endpoints of C. We also require that as i increases, P_i keeps moving in the same direction. We will approximate the area of sheet metal between points P_{i-1} and P_i as follows: we pretend $f(x, y)$ is constant on this piece of C, say, equal to $f(x_i^*, y_i^*)$, for some (x_i^*, y_i^*) between P_{i-1}, P_i, and we also pretend that C a straight line from P_{i-1} to P_i. With both of these approximations in mind, the surface area of the metal from P_{i-1} to P_i is given by

$$f(x_i^*, y_i^*)\sqrt{\Delta x_i^2 + \Delta y_i^2}$$

where Δx_i is the change in the x coordinate from P_{i-1} to P_i. Now, each of these points P_i is equal to $(x(t_i), y(t_i))$, where we assume $x(t), y(t), C$ are given in such a way to ensure that t_i are an increasing sequence of numbers, with $t_0 = a, t_n = b$. Therefore, if we sum all these approximations, we end up with a Riemann sum

$$\sum_i f(x(t_i^*), y(t_i^*))\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t.$$

The limit of this Riemann sum is defined to be the *line integral* of f on C, and is written

$$\int_C f(x, y)\, ds.$$

On the other hand, this Riemann sum looks like a Riemann sum for the variable t, and the limit as $\Delta t \to 0$ is equal to

$$\int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

If we write C using vector notation, and let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then this equation can be rewritten as

$$\int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\, dt.$$

Strictly speaking, to make sure that the line integral of $f(x, y)$ over C is equal to the expression we've given above, we want C to be smooth, which means that $\mathbf{r}'(t)$ exists and is everywhere nonzero, and $f(x, y)$ to be continuous.

Examples.

- Line integrals are genuine generalizations of definite integrals of a single variable. Suppose the curve C is an interval $[a, b]$ on the x-axis, and we have a function $f(x, 0) = f(x)$ defined on that interval. Then C can be parameterized by $x(t) = t, y(t) = 0$, where $a \le t \le b$. Then the line integral of f over C is equal to

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{1^2 + 0^2}\, dt = \int_a^b f(t, 0)\, dt = \int_a^b f(t)\, dt.$$

- Arc length integrals are special cases of line integrals. Suppose we want to evaluate the line integral of the constant function $f(x, y) = 1$ on C, parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then we have

$$\int_C f(x,y)\,ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt,$$

which is just the formula for the arc length of C.

- One consequence of the presence of the factor of $|\mathbf{r}'(t)|$ is that line integrals are independent of the parameterization of C, at least if we restrict ourselves to parameterizations which traverse each point of C exactly once. This is good to know, since our notation for line integrals depends only on $f(x,y)$ and C, and not the choice of parameterization for C. However, when actually calculating a line integral, you will need to determine some parameterization for C.

Example. Richard Serra has given you your first assignment! He wants to install a piece of sheet metal over the curve given by the part of $y = x^2$ between $x = 0, x = 2$, and the height of this metal at (x, x^2) is given by the function $f(x,y) = f(x,x^2) = 2x$. What is the surface area of the metal you must cut out for this installation?

When starting out with a line integral problem, the very first place to start is to determine a parameterization for C, if one is not given to you. In this case, we need to parameterize the part of $y = x^2$ which lies between $x = 0$ and $x = 2$. Perhaps the most obvious parameterization for this curve is $x(t) = t, y(t) = t^2, 0 \leq t \leq 2$. Then the corresponding line integral is

$$\int_C f(x,y)\,ds = \int_0^2 2t\sqrt{1^2 + (2t)^2}\,dt.$$

This can be evaluated using a u-substitution; let $u = 4t^2 + 1, du = 8t\,dt$, so teh above integral is equal to

$$\frac{1}{4}\int_1^{17} \sqrt{u}\,du = \frac{1}{4}\frac{2u^{3/2}}{3}\Big|_1^{17} = \frac{17^{3/2} - 1}{6}.$$

We've seen one interpretation of the value of a line integral: as the area of the surface over the curve C with height $f(x,y)$. Another interpretation involves mass or charge. Suppose we have a thin wire (thin enough to be thought of as a one-dimensional object) in the shape of a curve C, with density $\rho(x,y)$ at a point (x,y). Then the mass of the wire is given by the line integral

$$m = \int_C \rho(x,y)\,ds.$$

Of course, the coordinates of the center of mass are given by

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds, \overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds,$$

with analogous formulas for moments of inertia, etc.

Example. Suppose a wire is in the shape of the semicircle given by $x^2 + y^2 = 1, y \geq 0$, and has uniform density. What is the center of mass of the wire?

The x-coordinate is clearly 0 by symmetry. The y coordinate is something we have to calculate, though. Assume $\rho(x,y) = 1$. Then the mass of the wire is given by the arc length of C, which is evidently equal to π. We also need to calculate

$$\int_C y\,ds.$$

We can parameterize C by using $x(t) = \cos t, y(t) = \sin t, 0 \leq t \leq \pi$. Then this line integral is equal to

$$\int_0^\pi y(t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^\pi \sin t\,dt = 2.$$

Therefore, the y coordinate of the center of mass is equal to $2/\pi$.